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ON SOME FACTORIZATION FORMULAS OF K-k-SCHUR FUNCTIONS

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ABSTRACT. We give some new formulas about factorizations of K-k-Schur functions $g_\lambda^{(k)}$, analogous to the k-rectangle factorization formula $s^{(k)}_{(t^{k+1-t})\cup\lambda} = s^{(k)}_{(t^{k+1-t})} s_\lambda^{(k)}$ of k-Schur functions. Although the formula of the same form does not hold for K-k-Schur functions, we can prove that $g^{(k)}_{R_t}$ divides $g^{(k)}_{R_t\cup\lambda}$, and in fact more generally that $g_P^{(k)}$ divides $g^{(k)}_{P\cup\lambda}$ for any multiple k-rectangles P and any k-bounded partition λ. We give the factorization formula of such $g_P^{(k)}$ and the explicit formulas of $g^{(k)}_{P\cup\lambda}/g_P^{(k)}$ in some cases.

1. INTRODUCTION

Let k be a positive integer. *k-Schur functions* $s_\lambda^{(k)}$ and their K-theoretic analogues $g_\lambda^{(k)}$, which are called *K-k-Schur functions*, are symmetric functions parametrized by k-bounded partitions λ, namely by the weakly decreasing strictly positive integer sequences $\lambda = (\lambda_1, \dots, \lambda_l)$, $l \in \mathbb{Z}_{\geq 0}$, whose terms are all bounded by k.

Historically, k-Schur functions were first introduced by Lascoux, Lapointe and Morse [LLM03], and subsequent studies led to several (conjectually equivalent) characterizations of $s_\lambda^{(k)}$ such as the Pieri-like formula due to Lapointe and Morse [LM07], and Lam proved that k-Schur functions correspond to the Schubert basis of homology of the affine Grassmannian [Lam08]. Moreover it was shown by Lam and Shimozono that k-Schur functions play a central role in the explicit description of the Peterson isomorphism between quantum cohomology of the Grassmannian and homology of the affine Grassmannian up to suitable localizations [LS12].

These developments have analogues in K-theory. Lam, Schilling and Shimozono [LSS10] characterized the K-theoretic k-Schur functions as the Schubert basis of K-homology of the affine Grassmannian, and Morse [Mor12] investigated them from a conbinatorial viewpoint, giving their various properties including the Pieri-like formulas using affine set-valued strips (the form using cyclically decreasing words are also given in [LSS10]).

In this paper we start from this combinatorial characterization (see Definition 6) and show certain new factorization formulas of K-k-Schur functions.

Among the k-bounded partitions, those of the form $(t^{k+1-t}) = (\underbrace{t, \dots, t}_{k+1-t} =: R_t)$, $1 \leq t \leq k$, called *k-rectangle*, play a special role. In particular, if a k-bounded partition has the form $R_t \cup \lambda$, where the symbol $\cup$ denotes the operation of concatenating the two sequences and reordering the terms in the weakly decreasing order,

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then the corresponding k-Schur function has the following factorization property [LM07, Theorem 40]:

$$s^{(k)}_{R_t\cup\lambda} = s^{(k)}_{R_t} s^{(k)}_{\lambda}. \tag{1}$$

T. Ikeda suggested that the K-k-Schur functions should also possess similar properties, including the divisibility of $g^{(k)}_{R_t\cup\lambda}$ by $g^{(k)}_{R_t}$, and that it should be interesting to explore such properties. The present work is an attempt to materialize his suggestion.

We do show that $g^{(k)}_{R_t}$ divides $g^{(k)}_{R_t\cup\lambda}$ in the ring $\Lambda^{(k)} = \mathbb{Z}[h_1,\dots,h_k]$, where h_i denotes the complete homogeneous symmetric functions of degree i, of which the K-k-Schur functions form a basis. However, unlike the case of k-Schur functions, the quotient $g^{(k)}_{R_t\cup\lambda}/g^{(k)}_{R_t}$ is not a single term $g^{(k)}_{\lambda}$ but, in general, a linear combination of K-k-Schur functions with leading term $g^{(k)}_{\lambda}$, namely in which $g^{(k)}_{\lambda}$ is the only highest degree term. Even the simplest case where λ consists of a single part (r), $1\le r\le k$, displays this phenomenon: we show that

$$g^{(k)}_{R_t\cup(r)} = \begin{cases} g^{(k)}_{R_t}\cdot g^{(k)}_{(r)} & (\text{if } t<r),\\ g^{(k)}_{R_t}\cdot \left(g^{(k)}_{(r)} + g^{(k)}_{(r-1)} + \cdots + g^{(k)}_{\varnothing}\right) & (\text{if } t\ge r) \end{cases} \tag{2}$$

(actually we have $g^{(k)}_{(s)} = h_s$ for $1\le s\le k$, and $g^{(k)}_{\varnothing} = h_0 = 1$). So we may ask:

Question 1. *Which $g^{(k)}_{\mu}$, besides $g^{(k)}_{\lambda}$, appear in the quotient $g^{(k)}_{R_t\cup\lambda}/g^{(k)}_{R_t}$? With what cofficients?*

A k-bounded partition can always be written in the form $R_{t_1}\cup\cdots\cup R_{t_m}\cup\lambda$ with λ not having so many repetitions of any part as to form a k-rectangle. In such an expression we temporarily call λ the remainder. Proceeding in the direction of Question 1, one ultimate goal may be to give a factorization formula in terms of the k-rectangles and the remainder. In the case of k-Schur functions, the straightforward factorization in (1) above leads to the formula $s^{(k)}_{R_{t_1}\cup\cdots\cup R_{t_m}\cup\lambda} = s^{(k)}_{R_{t_1}}\cdots s^{(k)}_{R_{t_m}} g^{(k)}_{\lambda}$. On the contrary, with K-k-Schur functions, the simplest case having a multiple k-rectangle gives

$$g^{(k)}_{R_t\cup R_t} = g^{(k)}_{R_t}\sum_{\lambda\subset R_t} g^{(k)}_{\lambda}. \tag{3}$$

Hence we cannot expect $g^{(k)}_{R_t\cup R_t}$ to be divisible by $g^{(k)}_{R_t}$ twice. Instead, upon organizing the part consisting of k-rectangles in the form $R^{a_1}_{t_1}\cup\cdots\cup R^{a_m}_{t_m}$ with $t_1<\cdots<t_m$ and $a_i\ge 1$ $(1\le i\le m)$, with $R^a_t = \underbrace{R_t\cup\cdots\cup R_t}_{a}$, actually we show that

$$g^{(k)}_{R^{a_1}_{t_1}\cup\cdots\cup R^{a_m}_{t_m}\cup\lambda} \text{ is divisible by } g^{(k)}_{R^{a_1}_{t_1}\cup\cdots\cup R^{a_m}_{t_m}},$$

which actually holds whether or not λ is the remainder. Then we can subdivide our goal as follows:

Question 1′. *Which $g^{(k)}_{\mu}$, besides $g^{(k)}_{\lambda}$, appear in the quotient $g^{(k)}_{P\cup\lambda}/g^{(k)}_{P}$ where $P = R^{a_1}_{t_1}\cup\cdots\cup R^{a_m}_{t_m}$, and with what coefficients?*

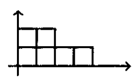
Question 2. *How can $g^{(k)}_{R^{a_1}_{t_1}\cup\cdots\cup R^{a_m}_{t_m}}$ be factorized?*

In this paper, we give a reasonably complete answer to Question 2 (Theorem 12), and partial answers to Question 1′ (Theorem 13, 14 and 15). The full paper will be published elsewhere.

2. Preliminaries

In this section we review some requisite combinatorial backgrounds. First recall that the Pieri rule characterizes Schur functions. In the theory of (K-)k-Schur functions, the underlying combinatorial objects are the set of *k-bounded partitions* (instead of partitions), which is isomorphic to the set of $(k+1)$-*cores*, and we have to consider *weak strips* instead of horizontal strips. For detailed definitions, see for instance [LLM+14, Chapter 2] or [Mac95, Chapter I].

2.1. **Partitions and Schur functions.** Let $\mathcal{P}$ denote the set of partitions. A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \in \mathcal{P}$ is identified with its *Young diagram* (or *shape*), for which we use the French notation here.



the Young diagram of $(4,2)$

We denote the *size* of a partition λ by $|\lambda|$, the *length* by $l(\lambda)$, and the *conjugate* by λ'. For partitions λ, μ we say $\lambda \subset \mu$ if $\lambda_i \leq \mu_i$ for all i. For a partition λ and a cell $c = (i,j)$ in λ, we denote the *hook length* of c in λ by $\mathrm{hook}_c(\lambda) = \lambda_i + \lambda'_j - i - j + 1$.

For a partition λ, a *removable corner* of λ (or λ-*removable corner*) is a cell $(i,j) \in \lambda$ with $(i,j+1),(i+1,j) \notin \lambda$. $(i,j) \in (\mathbb{Z}_{>0})^2 \setminus \lambda$ is said to be an *addable corner* of λ (or λ-*addable corner*) if $(i,j-1),(i-1,j) \in \lambda$ with the understanding that $(0,j),(j,0) \in \lambda$.

Let $\Lambda = \mathbb{Z}[h_1, h_2, \dots]$ be the ring of symmetric functions, generated by the *complete symmetric functions* $h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}$.

The *Schur functions* $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ are the family of symmetric functions satisfying the *Pieri rule*: $h_r s_\lambda = \sum s_\mu$, summed over μ such that μ/λ is a horizontal r-strip.

2.2. **Bounded partitions, cores and k-rectangles R_t.** A partition λ is called *k-bounded* if $\lambda_1 \leq k$. Let $\mathcal{P}_k$ be the set of all k-bounded partitions. An *r-core* (or simply a *core* if no confusion can arise) is a partition none of whose cells have a hook length equal to r. We denote by $\mathcal{C}_r$ the set of all r-core partitions.

Hereafter we fix a positive integer k.

For a cell $c = (i,j)$, the *content* of c is $j - i$ and the *residue* of c is $\mathrm{res}(c) = j - i \mod (k+1) \in \mathbb{Z}/(k+1)$. For a set X of cells, we write $\mathrm{Res}(X) = \{\mathrm{res}(c) \mid c \in X\}$. We will write a λ-removable corner of residue i simply a λ-removable i-corner. For simplicity of notation, we may use an integer to represent a residue, omitting "mod $(k+1)$".

We denote by R_t the partition $(t^{k+1-t}) = (t, t, \dots, t) \in \mathcal{P}_k$ for $1 \leq t \leq k$, which is called a *k-rectangle.* Naturally a k-rectangle is a $(k+1)$-core.

Now we recall the bijection between the k-bounded partitions in $\mathcal{P}_k$ and the $(k+1)$-cores in $\mathcal{C}_{k+1}$: The map $\mathfrak{p}\colon \mathcal{C}_{k+1} \longrightarrow \mathcal{P}_k; \kappa \mapsto \lambda$ is defined by $\lambda_i = \#\{j \mid (i,j) \in \kappa,\ \mathrm{hook}_{(i,j)}(\kappa) \leq k\}$. Then in fact $\mathfrak{p}$ is bijective and we put $\mathfrak{c} = \mathfrak{p}^{-1}$. See [LM05, Theorem 7] for details. Note that if λ is contained in a k-rectangle then $\lambda \in \mathcal{P}_k$ and $\lambda \in \mathcal{C}_{k+1}$, and besides $\mathfrak{p}(\lambda) = \lambda = \mathfrak{c}(\lambda)$.

For $i = 0, 1, \ldots, k$, an action s_i on $\mathcal{C}_{k+1}$ is defined as follows: For $\kappa \in \mathcal{C}_{k+1}$,

- if there is a κ-addable i-corner, then let $s_i \cdot \kappa$ be κ with all κ-addable i-corners added,
- if there is a κ-removable i-corner, then let $s_i \cdot \kappa$ be κ with all κ-removable i-corners removed,
- otherwise, let $s_i \cdot \kappa$ be κ.

In fact first and second case never occur simultaneously and $s_i \cdot \kappa \in \mathcal{C}_{k+1}$.

2.3. **Weak order and weak strips.** We review the weak order on $\mathcal{C}_{k+1}$.

Definition 1. *The weak order $\prec$ on $\mathcal{C}_{k+1}$ is defined by the following covering relation:*

$$\tau \prec\cdot\, \kappa \iff \exists i \text{ such that } s_i\tau = \kappa,\ \tau \subsetneq \kappa.$$

Definition 2. *For $(k+1)$-cores $\tau \subset \kappa \in \mathcal{C}_{k+1}$, κ/τ is called a weak strip of size r (or a weak r-strip) when*

$$\kappa/\tau \text{ is horizontal strip and } \tau \prec\cdot\, \exists\tau^{(1)} \prec\cdot\, \ldots \prec\cdot\, \exists\tau^{(r)} = \kappa.$$

2.4. **k-Schur functions.** We recall a characterization of k-Schur functions given in [LM07], since it is a model for and has a relationship with K-k-Schur functions.

Definition 3 (k-Schur function via "weak Pieri rule"). *k-Schur functions $\{s_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ are the family of symmetric functions such that $s_\varnothing^{(k)} = 1$ and*

$$h_r s_\lambda^{(k)} = \sum_\mu s_\mu^{(k)} \quad \text{for } r \leq k \text{ and } \mu \in \mathcal{P}_k,$$

summed over $\mu \in \mathcal{P}_k$ such that $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ is a weak strip of size r.

In fact $\{s_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ forms a basis of $\Lambda^{(k)} = \mathbb{Z}[h_1, \ldots, h_k] \subset \Lambda$. In addition $s_\lambda^{(k)}$ is homogeneous of degree $|\lambda|$. It is proved in [LM07, Theorem 40] that

Proposition 4 (k-rectangle property). *For $1 \leq t \leq k$ and $\lambda \in \mathcal{P}_k$, we have $s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)} (= s_{R_t} s_\lambda^{(k)})$.*

2.5. **K-k-Schur functions $g_\lambda^{(k)}$.** In [Mor12] a combinatorial characterization of K-k-Schur functions is given via an analogue of the Pieri rule, using some kind of strips called *affine set-valued strips.*

For a partition λ, $(i, j) \in (\mathbb{Z}_{>0})^2$ is called *λ-blocked* if $(i+1, j) \in \lambda$.

Definition 5 (affine set-valued strip). *For $r \leq k$, $(\gamma/\beta, \rho)$ is called an affine set-valued strip of size r (or an affine set-valued r-strip) if ρ is a partition and $\beta \subset \gamma$ are cores both containing ρ such that*

1. *γ/β is a weak $(r-m)$-strip where we put $m = \#\mathrm{Res}(\beta/\rho)$,*
2. *β/ρ is a subset of β-removable corners,*
3. *γ/ρ is a horizontal strip,*
4. *For $\forall i \in \mathrm{Res}(\beta/\rho)$, all β-removable i-corners which are not γ-blocked are in β/ρ.*

In this paper we employ the following characterization [Mor12, Theorem 48] of the K-k-Schur function as its definition.

Definition 6 (K-k-Schur function via an "affine set-valued" Pieri rule). *K-k-Schur functions $\{g_\lambda^{(k)}\}_{\lambda\in\mathcal{P}_k}$ are the family of symmetric functions such that $g_\varnothing^{(k)}=1$ and for $\lambda\in\mathcal{P}_k$ and $0\le r\le k$,*

$$h_r\cdot g_\lambda^{(k)}=\sum_{(\mu,\rho)}(-1)^{|\lambda|+r-|\mu|}g_\mu^{(k)}, \tag{4}$$

summed over (μ,ρ) such that $(\mathfrak{c}(\mu)/\mathfrak{c}(\lambda),\rho)$ is an affine set-valued strip of size r.

In fact $\{g_\lambda^{(k)}\}_{\lambda\in\mathcal{P}_k}$ forms a basis of $\Lambda^{(k)}$. Moreover, though $g_\lambda^{(k)}$ is an inhomogeneous symmetric function in general, the degree of $g_\lambda^{(k)}$ is $|\lambda|$ and its homogeneous part of highest degree is equal to $s_\lambda^{(k)}$.

3. Results

3.1. Possibility of factoring out $g^{(k)}_{R_{t_1}^{a_1}\cup\cdots\cup R_{t_m}^{a_m}}$ and some other general results. As discussed above, it does not hold that $g^{(k)}_{R_t\cup\lambda}=g^{(k)}_{R_t}g^{(k)}_\lambda$ for any $\lambda\in\mathcal{P}_k$. However, it holds that $g^{(k)}_{R_t}$ divides $g^{(k)}_{R_t\cup\lambda}$. We prove it in a slightly more general form.

The following notation is often referred later:

(NP) Let $1\le t_1,\dots,t_m\le k$ be distinct integers and $a_i\in\mathbb{Z}_{>0}$ $(1\le i\le m)$, where $m\in\mathbb{Z}_{>0}$. Then we put

$$P=R_{t_1}^{a_1}\cup\cdots\cup R_{t_m}^{a_m},$$
$$\alpha_P(u)=\#\{t_v\mid 1\le v\le m,\ t_v\ge u\}\quad\text{for each } u\in\mathbb{Z}_{>0}.$$

Proposition 7. *Let P be as in the above* (NP). *Then, for $\lambda=(\lambda_1,\cdots,\lambda_l)\in\mathcal{P}_k$, we have $g_P^{(k)}|g^{(k)}_{\lambda\cup P}$ in the ring $\Lambda^{(k)}$.*

Remark. Note that λ may still have the form $\lambda=R_t\cup\mu$. Hereafter we will not repeat the same remark in similar statements.

Since the homogeneous part of highest degree of $g_\lambda^{(k)}$ is equal to $s_\lambda^{(k)}$ for any λ, it follows from Propositions 4 and 7 that

Corollary 8. *Let P be as in* (NP). *Then, for any $\lambda\in\mathcal{P}_k$, we can write*

$$g^{(k)}_{P\cup\lambda}=g^{(k)}_P\left(g^{(k)}_\lambda+\sum_\mu a_{\lambda\mu}g^{(k)}_\mu\right),$$

summing over $\mu\in\mathcal{P}_k$ such that $|\mu|<|\lambda|$, for some coefficients $a_{\lambda\mu}$ (depending on P).

Now we are interested in finding a explicit description of $g^{(k)}_{P\cup\lambda}/g^{(k)}_P$. Let us consider the case $P=R_t$ for simplicity.

As noted above, a linear map Θ extending $g^{(k)}_\lambda\mapsto g^{(k)}_{R_t\cup\lambda}$ $(\forall\lambda\in\mathcal{P}_k)$ does not coincide with the multiplication of $g^{(k)}_{R_t}$ because it does not commute with the multiplication by h_r in the first place.

However, we can prove that the restriction of Θ to the subspace spanned by $\{g^{(k)}_{R_t\cup\mu}\}_{\mu\in\mathcal{P}_k}$ (in fact this is the principal ideal generated by $g^{(k)}_{R_t}$) commutes with the multiplication by h_r, and thus it coincides with the multiplication of $\Theta(g^{(k)}_{R_t})/g^{(k)}_{R_t}=$

$g^{(k)}_{R_t \cup R_t}/g^{(k)}_{R_t}$ on that ideal (Proposition 9). Thus it is of interest to describe the value of $g^{(k)}_{R_t \cup R_t}/g^{(k)}_{R_t}$, which is shown to be $\sum_{\nu \subset R_t} g^{(k)}_\nu$ later.

Proposition 9. *For $\lambda \in \mathcal{P}_k$ and $1 \le t \le k$, we have* $g^{(k)}_{\lambda \cup R_t \cup R_t} = g^{(k)}_{\lambda \cup R_t} \cdot \dfrac{g^{(k)}_{R_t \cup R_t}}{g^{(k)}_{R_t}}$.

As a corollary, it turns out that the value of $g^{(k)}_{P \cup \lambda}/g^{(k)}_P$ is independent of $a_1, \ldots, a_m$, the "multiplicities" of k-rectangles.

Theorem 10. *Let $P = R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m}$ be as in* (NP), *and put $Q = R_{t_1} \cup \cdots \cup R_{t_m}$. Then, for $\lambda \in \mathcal{P}_k$ we have*

$$\frac{g^{(k)}_{P \cup \lambda}}{g^{(k)}_P} = \frac{g^{(k)}_{Q \cup \lambda}}{g^{(k)}_Q}.$$

Thus we can reduce Question 1′ to the case where the k-rectangles are of all different sizes.

3.2. **Answer to Question** 2. For Question 2, we first show that multiple k-rectangles of different sizes entirely split, namely,

Theorem 11. *For $1 \le t_1 < \cdots < t_m \le k$ and $a_1, \ldots, a_m > 0$,*

$$g^{(k)}_{R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m}} = g^{(k)}_{R^{a_1}_{t_1}} \cdots g^{(k)}_{R^{a_m}_{t_m}}.$$

Then we show that for each $1 \le t \le k$ and $a > 1$, we have a nice factorization generalizing the formula (3):

Theorem 12. *For $1 \le t \le k$ and $a > 0$, we have*

$$g^{(k)}_{R^a_t} = g^{(k)}_{R_t} \left(\sum_{\lambda \subset R_t} g^{(k)}_\lambda \right)^{a-1}.$$

Thus, substituting this into Theorem 11, we have

$$g^{(k)}_{R^{a_1}_{t_1} \cup \cdots \cup R^{a_n}_{t_n}} = g^{(k)}_{R_{t_1}} \left(\sum_{\lambda^{(1)} \subset R_{t_1}} g^{(k)}_{\lambda^{(1)}} \right)^{a_1 - 1} \cdots g^{(k)}_{R_{t_n}} \left(\sum_{\lambda^{(n)} \subset R_{t_n}} g^{(k)}_{\lambda^{(n)}} \right)^{a_n - 1}.$$

3.3. **(Partial) Answer to Question 1′.** An easiest case is where $\lambda = (r)$ consists of a single part, which generalizes the case (2) in Introduction. Namely we show that

Theorem 13. *Let $P, \alpha_P(u)$ be as in* (NP)*and $1 \le r \le k$. Then we have*

$$\frac{g^{(k)}_{P \cup (r)}}{g^{(k)}_P} = \sum_{s=0}^{r} \binom{\alpha_P(r) + r - s - 1}{r - s} h_s.$$

In particular, if $t_m < r$, which means $\alpha_P(r) = 0$, we have

$$\frac{g^{(k)}_{P \cup (r)}}{g^{(k)}_P} = h_r = g^{(k)}_{(r)}$$

On the other hand, when $m = 1$,

$$\frac{g^{(k)}_{R_t \cup (r)}}{g^{(k)}_{R_t}} = \begin{cases} h_r & (\text{if } r > t), \\ h_r + h_{r-1} + \cdots + h_0 & (\text{if } r \leq t). \end{cases}$$

Then generalizing this case, we derive explicit formulas in the cases where $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfies the following condition (Nλ)and that the parts of λ except for λ_l are all larger than the widths of the k-rectangles.

(Nλ) Let $(\varnothing \neq)\lambda \in \mathcal{P}_k$ with satisfying $\bar{\lambda} \subset R'_{\bar{l}}$, where we write $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)-1})$, $l = l(\lambda)$ and $\bar{l} = l(\bar{\lambda}) = l - 1$. (Here we consider R_t to be $\varnothing$ unless $1 \leq t \leq k$)

(*Note*: when $l(\lambda) = 1$, we have $\bar{l} = 0$ and $\bar{\lambda} = \varnothing = R'_{\bar{l}}$ thus λ satisfies (Nλ). When $l(\lambda) > k+1$, we have $\bar{l} > k$ and $\bar{\lambda} \neq \varnothing = R'_{\bar{l}}$ thus λ does not satisfy (Nλ).)

Namely, we prove that

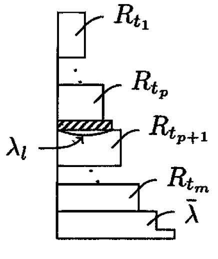
Theorem 14. *Let P and $\alpha_P(u)$ (for $u \in \mathbb{Z}_{>0}$) be as in* (NP)*in Section 3.1, before Proposition 7. Let $\lambda, l, \bar{\lambda}, \bar{l}$ be as in* (Nλ)*above. Assume* $\max_i\{t_i\} < \bar{\lambda}_{\bar{l}}$. *Then we have*

$$g^{(k)}_P g^{(k)}_\lambda = \sum_{s=0}^{\lambda_l} (-1)^s \binom{\alpha_P(\lambda_l + 1 - s)}{s} g^{(k)}_{P \cup \bar{\lambda} \cup (\lambda_l - s)}. \tag{1}$$

$$g^{(k)}_{P \cup \lambda} = g^{(k)}_P \sum_{s=0}^{\lambda_l} \binom{\alpha_P(\lambda_l) + s - 1}{s} g^{(k)}_{\bar{\lambda} \cup (\lambda_l - s)}. \tag{2}$$

In particular, if $t_n < \lambda_l$ then $\alpha_P(\lambda_l) = 0$ and

$$g^{(k)}_{P \cup \lambda} = g^{(k)}_P g^{(k)}_\lambda.$$



In this figure $p = m - \alpha_P(\lambda_l)$ and $a_i = 1$ for all i.

Moreover, we show a formula in a slightly different case where P is a single k-rectangle R_t and $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfies (Nλ)and that the parts of λ except for λ_l are all larger than *or equal to* the widths of the k-rectangles.

Notation. For any partition λ, let $\lambda^\circ = (\lambda_1, \dots, \lambda_i)$ if $\lambda_i > t \geq \lambda_{i+1}$ (we set $\lambda^\circ = \varnothing$ if $t \geq \lambda_1$).

Theorem 15. *Let $\lambda, l, \bar{\lambda}, \bar{l}$ be as in* (Nλ). *Assume $\bar{\lambda}_{\bar{l}} \geq t \geq \lambda_l$. Then we have*

$$g^{(k)}_{R_t \cup \lambda} = g^{(k)}_{R_t} \sum_{\mathfrak{c}(\lambda^\circ) \subset \mathfrak{c}(\nu) \subset \mathfrak{c}(\lambda)} g^{(k)}_\nu.$$

3.4. Example. Let us illustrate the sketch of the proof of Theorem 14 with a small example.

Consider the case $P = R_t$: we shall show that $g^{(k)}_{R_t \cup \lambda} = g^{(k)}_{R_t} g^{(k)}_{\lambda}$ if $\lambda_l > t$ and $\lambda_1 + l \le k + 2$. Let us assume Theorem 13 (the case $l = 1$) and consider the case $l = 2$. Set $\lambda = (a, b)$ with $k \ge a \ge b > t$.

Step (A): Expand $g^{(k)}_{(a,b)}$ into a linear combination of products of complete symmetric functions and K-k-Schur functions labeled by partitions with fewer rows:

By using the Pieri rule (4) we have

$$\begin{aligned} g^{(k)}_{(a)} h_i = &\left(g^{(k)}_{(a,i)} - g^{(k)}_{(a,i-1)}\right) \\ &+ \left(g^{(k)}_{(a+1,i-1)} - g^{(k)}_{(a+1,i-2)}\right) \\ &+ \dots \\ &\begin{cases} \dots + \left(g^{(k)}_{(a+i-1,1)} - g^{(k)}_{(a+i-1,0)}\right) & \\ \quad + g^{(k)}_{(a+i,0)} & (\text{if } a+i \le k) \\ \dots + \left(g^{(k)}_{(k-1,a+i-k+1)} - g^{(k)}_{(k-1,a+i-k)}\right) & \\ \quad + \left(g^{(k)}_{(k,a+i-k)} - g^{(k)}_{(k,a+i-k-1)}\right) & (\text{if } a+i > k) \end{cases} \end{aligned}$$

for $i \le a$, and summing this over $0 \le i \le b$, we have

$$\begin{aligned} g^{(k)}_{(a)} (h_b + \dots + h_0) &= g^{(k)}_{(a,b)} + g^{(k)}_{(a+1,b-1)} + \dots \begin{cases} g^{(k)}_{(a+b,0)} & (\text{if } a+b \le k) \\ g^{(k)}_{(k,a+b-k)} & (\text{if } a+b \ge k) \end{cases} \\ &= \sum_{\substack{\mu/(a):\text{horizontal strip} \\ |\mu|=a+b \\ \mu_1 \le k}} g^{(k)}_{\mu}. \end{aligned} \tag{5}$$

Similarly we have

$$g^{(k)}_{(a+1)} (h_{b-1} + \dots + h_0) = g^{(k)}_{(a+1,b-1)} + g^{(k)}_{(a+2,b-2)} + \dots = \sum_{\substack{\mu/(a+1):\text{horizontal strip} \\ |\mu|=a+b \\ \mu_1 \le k}} g^{(k)}_{\mu},$$

hence

$$g^{(k)}_{(a,b)} = g^{(k)}_{(a)} (h_b + \dots + h_0) - g^{(k)}_{(a+1)} (h_{b-1} + \dots + h_0).$$

Step (B): Multiply $g^{(k)}_{(a,b)}$ by $g^{(k)}_{R_t}$. Then we have

$$\begin{aligned} g^{(k)}_{R_t} g^{(k)}_{(a,b)} &= g^{(k)}_{R_t} g^{(k)}_{(a)} (h_b + \dots + h_0) - g^{(k)}_{R_t} g^{(k)}_{(a+1)} (h_{b-1} + \dots + h_0) \\ &= g^{(k)}_{R_t \cup (a)} (h_b + \dots + h_0) - g^{(k)}_{R_t \cup (a+1)} (h_{b-1} + \dots + h_0) \end{aligned}$$

because $g^{(k)}_{R_t} g^{(k)}_{(a)} = g^{(k)}_{R_t \cup (a)}$ since $t < a$.

Then carry out calculations similar to Step (A).

Notation. For a proposition P, we shall write $\delta[P] = 1$ if P is true and $\delta[P] = 0$ if P is false.

Since the number of residues of $\mathfrak{c}(R_t \cup (a,j))$-nonblocked $\mathfrak{c}(R_t \cup (a))$-removable corners is $1 + \delta[t > j]$,

$$\begin{aligned} g^{(k)}_{R_t \cup (a)} h_i = &\left(g^{(k)}_{R_t \cup (a,i)} - \binom{1 + \delta[t > i-1]}{1} g^{(k)}_{R_t \cup (a,i-1)} + \binom{1 + \delta[t > i-2]}{2} g^{(k)}_{R_t \cup (a,i-2)} \right) \\ &+ \left(g^{(k)}_{R_t \cup (a+1,i-1)} - \binom{1 + \delta[t > i-2]}{1} g^{(k)}_{R_t \cup (a+1,i-2)} + \binom{1 + \delta[t > i-3]}{2} g^{(k)}_{R_t \cup (a+1,i-3)} \right) \end{aligned}$$

$+\dots.$

Summing this over $0 \le i \le b$, we have

$$\begin{aligned} g^{(k)}_{R_t\cup(a)}(h_b+\dots+h_0) &= \left(g^{(k)}_{R_t\cup(a,b)} - \delta\,[t>b-1]\,g^{(k)}_{R_t\cup(a,b-1)}\right) \\ &\quad + \left(g^{(k)}_{R_t\cup(a+1,b-1)} - \delta\,[t>b-2]\,g^{(k)}_{R_t\cup(a+1,b-2)}\right) \\ &\quad + \dots. \end{aligned}$$

Similarly we have

$$\begin{aligned} g^{(k)}_{R_t\cup(a+1)}(h_{b-1}+\dots+h_0) &= \left(g^{(k)}_{R_t\cup(a+1,b-1)} - \delta\,[t>b-2]\,g^{(k)}_{R_t\cup(a+1,b-2)}\right) \\ &\quad + \left(g^{(k)}_{R_t\cup(a+2,b-2)} - \delta\,[t>b-3]\,g^{(k)}_{R_t\cup(a+2,b-3)}\right) \\ &\quad + \dots, \end{aligned}$$

hence we have

$$\begin{aligned} g^{(k)}_{R_t}g^{(k)}_{(a,b)} &= g^{(k)}_{R_t\cup(a,b)} - \delta\,[t>b-1]\,g^{(k)}_{R_t\cup(a,b-1)} \\ &= g^{(k)}_{R_t\cup(a,b)} \end{aligned}$$

since we have assumed $b>t$.

4. Discussions

It is worth noting that, in any cases we have seen, $g^{(k)}_{P\cup\lambda}/g^{(k)}_P$ is written as a linear combination with *positive coefficients* of K-k-Schur functions:

Conjecture 16. *For* $\forall\lambda\in\mathcal{P}_k$ *and* $P=R^{a_1}_{t_1}\cup\dots\cup R^{a_m}_{t_m}$, *write*

$$g^{(k)}_{P\cup\lambda} = g^{(k)}_P\sum_\mu a_{P,\lambda,\mu}g^{(k)}_\mu.$$

Then $a_{P,\lambda,\mu}\ge 0$ *for any* μ.

In the case $P=R_t$, it is observed that $a_{R_t,\lambda,\mu}=0$ or 1. Moreover, the set of μ such that $a_{R_t,\lambda,\mu}=1$ is expected to be an "interval", but we have to consider the *strong order* on $\mathcal{P}_k\simeq\mathcal{C}_{k+1}$, which can be seen as just inclusion as shapes in the poset of cores. Namely, the strong order $\lambda\le\mu$ on $\mathcal{P}_k$ is defined by $\mathfrak{c}(\lambda)\subset\mathfrak{c}(\mu)$. Notice that $\lambda\preceq\mu\Longrightarrow\lambda\subset\mu\Longrightarrow\lambda\le\mu$ for $\lambda,\mu\in\mathcal{P}_k$. Then,

Conjecture 17. *For* $\forall\lambda\in\mathcal{P}_k$ *and* $1\le\forall t\le k$, *there exists* $\exists\mu\in\mathcal{P}_k$ *such that*

$$g^{(k)}_{R_t\cup\lambda} = g^{(k)}_{R_t}\sum_{\mu\le\nu\le\lambda} g^{(k)}_\nu.$$

It should be interesting to study the geometric meaning of these results and conjectures.

References

[Lam08] Thomas Lam, *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. **21** (2008), no. 1, 259–281.

[LLM03] L. Lapointe, A. Lascoux, and J. Morse, *Tableau atoms and a new Macdonald positivity conjecture*, Duke Math. J. **116** (2003), no. 1, 103–146.

[Las01] Alain Lascoux, *Ordering the affine symmetric group*, Algebraic combinatorics and applications (Gößweinstein, 1999), Springer, Berlin, 2001, pp. 219–231.

[LLM+14] Thomas Lam, Luc Lapointe, Jennifer Morse, Anne Schilling, Mark Shimozono, and Mike Zabrocki, *k-Schur functions and affine Schubert calculus*, Fields Institute Monographs, vol. 33, Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2014.

[LLMS10] Thomas Lam, Luc Lapointe, Jennifer Morse, and Mark Shimozono, *Affine insertion and Pieri rules for the affine Grassmannian*, Mem. Amer. Math. Soc. **208** (2010), no. 977, xii+82.

[LM04] L. Lapointe and J. Morse, *Order ideals in weak subposets of Young's lattice and associated unimodality conjectures*, Ann. Comb. **8** (2004), no. 2, 197–219.

[LM05] Luc Lapointe and Jennifer Morse, *Tableaux on $k+1$-cores, reduced words for affine permutations, and k-Schur expansions*, J. Combin. Theory Ser. A **112** (2005), no. 1, 44–81.

[LM07] Luc Lapointe and Jennifer Morse, *A k-tableau characterization of k-Schur functions*, Adv. Math. **213** (2007), no. 1, 183–204.

[LS12] Thomas Lam and Mark Shimozono, *From quantum Schubert polynomials to k-Schur functions via the Toda lattice*, Math. Res. Lett. **19** (2012), no. 1, 81–93.

[LSS10] Thomas Lam, Anne Schilling, and Mark Shimozono, *K-theory Schubert calculus of the affine Grassmannian*, Compos. Math. **146** (2010), no. 4, 811–852.

[Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995.

[Mor12] Jennifer Morse, *Combinatorics of the K-theory of affine Grassmannians*, Adv. Math. **229** (2012), no. 5, 2950–2984.